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THE GROUPS WHICH CONTAIN LESS THAN TWENTY OPERATORS OF ORDER THREE.

By G. A. MILLER.

All the operators of a given order which are contained in a group (G) generate an invariant subgroup if they do not generate the entire group. As the existence of invariant subgroups in G is generally very useful in determining other properties, it is of interest to know what groups contain a given number of operators of a certain order and are generated by these operators. Very little has been done along this line. One of the most interesting problems in this connection is the determination of all the groups which contain a small number of operators of order 2 and are generated by these operators. This problem has been solved in case the number of the operators of order 2 does not exceed 15.

The present note is devoted to a study of a few of the groups which seem almost as interesting as those just mentioned. It will be observed that very few theorems of group theory are employed in these considerations. According to a well known theorem due to Frobenius, the number of subgroups of order p in any group is of the form $1+kp$. Hence the groups which come under the present heading contain 2, 8, or 14 operators of order 3. In the first case the operators of order 3 generate an invariant subgroup of this order. The other two cases lead to much more interesting results.

§ 1. *Groups which contain just eight operators of order three and are generated by these operators.*

Let s_1, s_2, s_3, s_4 represent the generators of the four subgroups of order 3 which are contained in G. If one of these four operators is transformed into itself by some other one, the eight operators are contained in a group of order 9. If this is not the case each of them transforms the other three and hence it transforms among themselves the subgroups which they generate. Hence G transforms these four subgroups according to a transitive substitution group (T) of degree 4. The subgroup (H) of G which corresponds to the identity in T cannot contain any operator of order 3. From this it follows that the order of T is divisible by 3. That is, T is either the alternating or the symmetric group of degre 4.

It is evident that there is a (1, 1) correspondence between the operators of order 3 in G and the substitutions of order 3 in T. We may therefore suppose that the product s_1s_2 corresponds to a substitution of order 2 in T. As s_1s_2 is commutative with every operator of H it has either three or six conjugates under G. In the former case the three conjugates of s_1s_2 with respect to its factors are identical. That is, s_1s_2, s_2s_1, $s_1^2s_2s_1^2$ are the same as s_1s_2, $s_2^2s_1s_2^2$, s_2s_1. Hence $s_1^2s_2s_1^2=s_2^2s_1s_2^2$. The former is the inverse of $s_1s_2.s_2s_1$[*] and the latter is the inverse of $s_2s_1.s_1s_2$. Since the inverses of these products are equal the products are equal and the three conjugates s_1s_2, s_2s_1, $s_1^2s_2s_1^2$ are commutative; for, if two of a complete set of three conjugates are commutative the three must be commutative.

We have now proved that the conjugates of s_1s_2 under G are commutative whenever s_1s_2 has only three such conjugates. It remains to prove that s_1s_2 is of order 2 in this case. From the equations $(s_1s_2.s_2s_1)^2=(s_1s_2)^4$ and $(s_1s_2.s_2s_1)^2=(s_1^2s_2s_1^2)^{-2}=(s_1s_2)^{-2}$ it follows that s_1s_2 is of order 6. The equation $(s_1^2s_2s_1^2)^{-2}$ follows from the fact that if an operator transforms the a power of another operator into itself the a power of the second operator is equal to the a power of the transform. As $(s_1s_2)^6=1$ and the order of H is not divisible by 3 it results that s_1s_2 is of order two.

Since s_1s_2, s_2s_1 are commutative and of order 2 they generate the four-group. As their product is its own inverse, it is $s_1^2s_2s_1^2$. In other words, the four group generated by s_1s_2, s_2s_1 is transformed into itself by s_1, and s_1 has four conjugates under this four-group. These results may be stated as follows:

If a group contains only four subgroups of order 3 and if the product of two of its non-commutative operators of order 3 has only three conjugates under the group then these four subgroups generate the alternating group of order 12.

It remains to consider the case when s_1s_2 has six conjugates under G. Two of the six conjugates correspond to the same substitution of T. These two are commutative and each of them transforms the remaining four, otherwise one of these six conjugates would be transformed into itself by more than one-sixth of the operators of G. As each of these six conjugates is commutative with just half of the operators of G which correspond to the four-group in T, it follows that the group generated by them has a commutator subgroup of order 2, which we shall represent by 1, c.

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[*] This is a special case of the theorem, If the product of two operators of order n is transformed successively by one of these operators, the continued product of the n conjugates thus obtained, in order, is the identity.

From the fact that $s_1s_2.s_2s_1$ is of the same order as $s_1^2s_2s_1^2$ and hence also of the same order as s_1s_2 it follows that the order of s_1s_2 is divisible by 4 but not by 8. In fact, $(s_1s_2.s_2s_1)^2=c(s_1s_2)^4$ is of the same order as $(s_1s_2)^2$ only when the order of s_1s_2 is divisible by 4 but not by 8. As $c(s_1s_2)^{-4}=(s_1^2s_2s_1^2)^2=(s_1s_2)^2$ it results that $c=(s_1s_2)^6$; and hence s_1s_2 is of order 4, the order of H being non-divisible by 3. The two conjugates s_1s_2, s_2s_1 must therefore generate the quaternion group. As this includes the inverse of $s_1^2s_2s_1^2$, it is transformed into itself by s_1. Moreover, s_1 has four conjugates under this group. These results may be expressed as follows:

If a group contains only four subgroups of order 3 the operators of this order generate the non-cyclic group of order 9, the alternating group of order 12, or the group of order 24 which does not include a subgroup of order 12.[*] *Hence it contains one of these subgroups invariantly.*

§ 2. *Groups which contain just fourteen operators of order three and are generated by these operators.*

If a group contains just fourteen operators of order 3 its Sylow subgroups must be cyclic since 7 is not of the form $1+p+kp^2$.[†] Hence the seven subgroups of order 3 form a single set of conjugates and are transformed by G according to a transitive group T of degree seven which contains just 7 subgroups of order 3. As in the preceding section there is evidently a (1, 1) correspondence between the operators of order 3 in G and the substitution of this order in T.

Two operators of order 3 in G, s_1, s_2 may be so selected that their product s_1s_2 corresponds to a substitution of order 7 in T. The three conjugates s_1s_2, s_2s_1, $s_1^2s_2s_1^2$ are commutative since T contains only one subgroup of order 7 and s_1s_2 is commutative with every operator of the subgroup H corresponding to the identity in T. From the equations $(s_1s_2.s_2s_1)^7=(s_1s_2)^{14}=(s_1^2s_2s_1^2)^{-7}=(s_1s_2)^{-7}$ it follows that $(s_1s_2)^{21}=1$. As H does not include any operator of order 3, s_1s_2 is of order 7.

The group generated by s_1s_2, s_2s_1 is either of order 7 or of order 49, since these operators are commutative. As this group includes $s_1^2s_2s_1^2$ it is transformed into itself by s_1. If its order is 7 the seven subgroups in question generate the group of order 21 which is defined by its order and the fact that it contains seven subgroups of order 3. As this group is so well known it remains only to prove that s_1s_2, s_2s_1 cannot generate a group of order 48. If this were the case s_1 would transform at least two of its 8 subgroups of order 7 into themselves. As it could not be commutative with the generators of both of these subgroups, it and one of these subgroups would again generate the given group of order 21. Hence it results that, *If a group contains just 14 operators of order 3 these operators generate the semi-metacyclic group of degree 7.*

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[*] *Quarterly Journal of Mathematics*, Vol. 28 (1896), p. 274.

[†] *Transactions of The American Mathematical Society*, Vol. 6 (1905), p. 58.